**SYNOPSIS ON**

# A STUDY OF UNIVALENT ANALYTIC FUNCTIONS ASSOCIATED WITH DISTRIBUTION SERIES

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# 1 Introduction

In this section, we shall review some of those aspects of the theory of special functions that are relevant for Geometric Function Theory, especially for conformal and quasi conformal maps. We restrict ourselves mainly to most important special functions, and briefly mention contexts where these functions occur. We do not attempt to give the results in their most general form, but rather our goal is to provide formulations which are easy to understand and adequate for many applications. One of the reasons for the frequent occurrence of special functions is that solutions of extremal problems can often be expressed in terms of special functions. Another reason is that some important conformal mappings are given by special functions. For instance, the conformal mapping of an annulus onto the complement of two closed segments on the real axis and the conformal mapping of a square onto a rectangle are given by elliptic functions. The study of conformal invariants also often leads to special functions.

In the 19th century, the definition of the gamma function was extended to complex numbers. It is well known that this function is very important in mathematics, physics and engineering. The hypergeometric function is one of the most important special functions, because of its many connections to other classes of special functions, and its numerous identities and expressions in terms of series and integrals. There is a considerable literature about the evaluation of hypergeometric functions for certain values of the argument in terms of other special functions such as the gamma function.

Functions that occur frequently, and are therefore useful, are often called special functions. In many cases, special functions are solutions of ordinary or partial differential equations or of extremal problems. Complete elliptic integrals arise in many fields of mathematics as well as in many physical problems.Theta functions play an important role in the study of elliptic functions and elliptic integrals. They have been extensively studied since the beginning of the 19th century, and they often occur in mathematical physics and also in pure mathematics.

As a result of their applications in solving various physical, biological, and engineering problems, the theory of special functions has a long and varied history with an enormous literature. There are special functions that arise from the solution of partial differential equations that satisfy certain conditions.A series, a generating function, an infinite product, an integral, a difference equation, a series of trigonometric or orthogonal functions. Eminent mathematicians notably Euler, Legendre, Gauss, Jacobi, Weierstrass, Kummer, Riemann, Ramanujan worked hard to develop special functions like Bessel functions, Whittaker functions, Gauss hypergeometric function and the polynomials that go by the names of Jacobi, Legendre, Laguerre, Hermite etc.

A lot of mathematics research is being done in the area of geometric function theory (GFT), is a sub-branch of complex analysis. The Study of GFT involves finding the relationship between the analytical properties

of the function of complex variable $f(z)$, and the geometrical properties of the image domain$D = f(E)$ where$E = \{z \in \mathbb{C} : |z| < 1\}$ 1ies the unit disc. We attempt to investigate some geometrical aspects of Univalent and Multivalent functions. We have obtained some nice results of coefficients estimates, convolution property, growth and distortion bounds, extreme points, criterion for univalency, $p$ -valency and extremal functions for these classes. The study will continue the investigation of several properties of analytic and univalent functions and the different subclasses like $S^*$- starlike, $C$- convex,$K$ -close-to-convex, uniformly starlike and convex, parabolic starlike and convex etc. We are also interested in the application of hypergeometric functions, fractional derivate operators, and fractional integral operators to various subclasses of polylogarithms as well as univalent and multivalent functions. Additionally being examined is the study of differential subordinations, univalency criteria, and their applications to distinct subclasses of both univalent and multivalent functions.Using GFT and the theory of complex analysis, we will try to find the connections with other branches of mathematics and applications in science and engineering.

Numerous subclasses of multivalent and univalent functions with numerous features will be defined. Geometric functions theory is one of the most significant subfields of complex analysis, which is one of mathematics' most remarkable fields. In particular, this work aims to determine correlations between geometrical and analytical aspects in the picture domain where the unit disc is located.

## 2 Objectives

We propose to study in the following areas:

1. To solve $q-$ analogus Meromorphic functions by using convolution properties.
2. To find radii properties and neighborhood results of analytic functions.
3. To find coefficient bounds related to Rabnotav, Miller Ross functions.
4. To solve differential operators by using subordination principles.
5. The study of q- generalized bessel functions of differential and integral operators and find out their possible new applications.
6. To study of q-analogues on various generalization of coefficient bounds of meromorphic Bi-univalent differential and integral operators.
7. To obtain partial sums and neighborhood properties of subclasses.
8. To obtain distortion bounds for the functions of these classes.
9. To find applications of harmonic functions of anlytic functions.

10. To find various applications of special polynomials like Gegangauer, Lucas, Pascal, Borel, Binomial distribution series, Negative Binomial distributions.

11. To study of q-analogues on various generalization of differential and integral operators of Bi-univalent functions.

12. Various applications of Geometric function Theory.

## 3 Importance of Study

Results obtained in this survey may be applied to solve problems in physical science, engineering, and statistics that arise from differential and integral equations. Our discussion included the following topics: the application of different polynomials and binomial, Pascal, Touchard, and other properties for the subclasses; basic properties, convolution properties, partial sums, weighted mean, arithmetic mean, linear combination, inclusion relation, and neighborhood properties.

The theory of subordination and the features of convolution are used to introduce a number of subclasses of meromorphic functions. These subclasses extend the classical subclasses of meromorphic starlikeness, convexity, close-to-convexity, and quasi-convexity. Each convoluted derived function in the class to be subordinated is subjected to a specific normalized convex function with positive real component. Investigations are conducted into class relations as well as the inclusion and convolution aspects of these subclasses.

It is anticipated that the majority of the survey's results will have use in the resolution of specific fractional order differential and integral equation issues that come up in the fields of statistics, engineering, and physical science. Borel distributions and summing in geometric function theory are a rich and significant field of study with applications in many different areas of theoretical physics and mathematics.

**International Status:** In 1907, Koebe [1] launched a global investigation into the theory of univalent functions. There are many subclasses of analytical and univalent functions. A novel and productive avenue of study in the theory of univalent functions was made possible by Silverman's proposal of a class $T$ of functions with negative coefficients. The study of a univalent function with negative coefficients led to a new line of inquiry into this theory: the creation of subclasses of analytic univalent functions in an effort to prove the validity of the well-known Bieberbach conjecture, which Branges positively settled in 1985.

**National Status:** In order to establish an inequality of the subclasses of univalent functions using the convolution method, Wanas and Khuttar [2] looked into a number of analytic function features related to the Borel operator. They have made an effort to determine the coefficient estimate, the hadamard product connected to the aforesaid operator, the distortion theorem, and extreme points.

## 4 Methodology

In the present study, it is proposed:

1. To investigate the generalized Mittag-Leffler type Borel distribution series.
2. To obtain some inclusion results for Geganbauer polynomials and Touchard polynomials.
3. To derive the recurrence relations and integral representations of the Rabnotav and Miller Ross type functions.
4. To deduce the results for the basic hypergeometric functions as an application of fractional $q-$ Calculus
5. To find neighborhood results.
6. To obtain partial sums properties to pascal distributation series.
7. To investigate coefficient bounds related negative Binomial distribution series.

## 5 Review of Literature

When Koebe found that all functions in $S$ have a common disc $|w| < b$, where $b$ is an absolute constant, in 1907, he laid the groundwork for the study of univalent functions. The Koebe function demonstrates that $b = 1/4$. Bieberbach [3] proved that $b$ can be taken to be $1/4$, and he also demonstrated that if S, then equality occurs only for rotations of the Koebe function. Because the Koebe function appears to be an external function for so many problems in class $S$, Bieberbach suspected that it maximises for every $n$, resulting in the famous Bieberbach conjecture, first proposed in 1916.

For many years, this well-known issue acted as a test, spurring the creation of significant ideas and creative methods that today form the cornerstone of the course.A number of advancements in the theory of univalent functions have been discussed by Bernardi [4], Hayman [5], and Goodman [6]. Duren has also written books on a few elements of the subject [7]. An outline of all the findings in convolution theory for univalent functions may be found in Rusheweyh's book [8]. Both the notion of convex functions of order and the concept of order of Starlikeness are introduced by Robertson [9]. The class of strongly Starlike functions, which was first presented by Brannan and Kirwan [10], is yet another subclass of Starlike functions that has drawn a lot of interest from professionals in the field recently.

## 6 Basic definitions

Let $A$ indicate the class of all functions $u(z)$ of the form

$$u(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{6.1}$$

in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$. Let $S$ be a subclass of $A$ that contains univalent functions and satisfies the usual normalization condition $u(0) = u'(0) - 1 = 0$. The subset of $A$ comprising of functions $u(z)$ that are all univalent in $E$ is represented by $S$. A function $u \in A$ is a starlike function of the order $\upsilon, 0 \leq \upsilon < 1$, if it fulfils

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \upsilon, (z \in E). \tag{6.2}$$

We indicate this class with $S^*(\upsilon)$ .

A function $u \in A$ is a convex function of the order $\upsilon, 0 \leq \upsilon < 1$, if it fulfils

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > \upsilon, (z \in E). \tag{6.3}$$

We indicate this class with $K(\upsilon)$.

The regular classes of starlike and convex functions in $E$ are $S^*(0) = S^*$ and $K(0) = K$, respectively. Let $T$ indicate the class of functions analytic in $E$ that are of the form

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0,\ z \in E) \tag{6.4}$$

and let $T^*(\upsilon) = T \cap S^*(\upsilon),\ C(\upsilon) = T \cap K(\upsilon)$. Silverman [11] has thoroughly studied the class $T^*(\upsilon)$ and related classes, which have some interesting properties.

For $u \in A$ given by (6.1) and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

their convolution indicate by $(u * g)$, is specified as

$$(u * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * u)(z), \quad (z \in E).$$

Note that $u * g \in A$.

The following subclasses were introduced and examined by Goodman [12,13] and Ronning [14,15] :

(1). If a function $u \in A$ satisfies the condition, it is said to be in the class $UCV(\varrho, \gamma)$, a uniformly $\gamma-$

convex function

$$\Re\left\{1+\frac{zu''(z)}{u'(z)}-\varrho\right\}>\gamma\left|\frac{zu''(z)}{u'(z)}\right|, \tag{6.5}$$

where $\gamma\geq 0, -1<\varrho\leq 1$ and $\varrho+\gamma\geq 0$.

(2). If a function $u\in A$ satisfies the condition, it is said to be in the class $SP(\varrho,\gamma)$, uniformly $\gamma-$ starlike function

$$\Re\left\{\frac{zu'(z)}{u(z)}-\varrho\right\}>\gamma\left|\frac{zu'(z)}{u(z)}-1\right|, \tag{6.6}$$

where $\gamma\geq 0, -1<\varrho\leq 1$ and $\varrho+\gamma\geq 0$.

## Mittag-Leffler function and Borel distribution

The study of operators is essential in geometric function theory, complex analysis, and other related fields. Convolution of certain analytic functions may be used to express a variety of derivative and integral operators. It should be highlighted that this formalism aids future mathematical study and a better understanding of the geometric properties of such operators. Let $E_\tau(z)$ and $E_{\tau,\nu}(z)$ be functions defined by

$$E_\tau(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\tau n+1)},\quad (z\in\mathbb{C},\Re(\tau)>0)$$

and

$$E_{\tau,\nu}(z)=\frac{1}{\Gamma(\nu)}+\sum_{n=1}^{\infty}\frac{z^n}{\Gamma(\tau n+\nu)},\quad (\tau,\nu\in\mathbb{C},\Re(\tau)>0,\Re(\nu)>0).$$

It can be written in other form

$$E_{\tau,\nu}(z)=\frac{1}{\Gamma(\nu)}+\sum_{n=2}^{\infty}\frac{z^{n-1}}{\Gamma(\tau(n-1)+\nu)},\quad (\tau,\nu\in\mathbb{C},\Re(\tau)>0,\Re(\nu)>0).$$

The function $E_\tau(z)$ was introduced by Mittag-Leffler Attiya[16], Banslal and Prajapat [17] and is, therefore, known as the Mittag-Leffler function. A more general function $E_{\tau,\nu}$ generalizing $E_\tau(z)$ was introduced by Wiman [18] and defined by

$$E_{\tau,\nu}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\tau n+\nu)},\quad (z,\tau,\nu\in\mathrm{C},\Re(\tau)>0,\Re(\nu)>0).$$

Observe that the function $E_{\tau,\nu}$ contains many well-known functions as its special case, for example,

$$E_{1,1}(z) = \mathrm{e}^z, \quad E_{1,2}(z) = \frac{\mathrm{e}^z - 1}{z},$$
$$E_{2,1}\left(z^2\right) = \cosh z, \quad E_{2,1}\left(-z^2\right) = \cos z, \quad E_{2,2}\left(z^2\right) = \frac{\sinh z}{z},$$
$$E_{2,2}\left(-z^2\right) = \frac{\sin z}{z}c, \quad E_3(z) = \frac{1}{2}\left[e^{z^{1/3}} + 2e^{-\frac{1}{2}z^{1/3}}\cos\left(\frac{\sqrt{3}}{2}z^{1/3}\right)\right]$$
$$\text{and } E_4(z) = \frac{1}{2}\left[\cos z^{1/4} + \cosh z^{1/4}\right].$$

The Mittag-Leffler function appears naturally in the solution of fractional order differential and integral equations. In the study of complex systems and super diffusive transport, in particular, fractional generalisation of the kinetic equation, random walks, and Levy flights. Several properties of Mittag-Leffler function and generalized Mittag-Leffler function can be found in [16,17]. Observe that MittagLeffler function $E_{\tau,\nu}(z)$ does not belong to the family $\mathcal{A}$. Thus, it is natural to consider the following normalization of Mittag-Leffler functions as below:

$$E_{\tau,\nu}(z) = z\Gamma(\nu)E_{\tau,\nu}(z) = z + \sum_{n=2}^{\infty}\frac{\Gamma(\nu)}{\Gamma(\tau(n-1)+\nu)}z^n, \tag{6.7}$$

it holds for complex parameters $\tau, \nu$ and $z \in \mathbb{C}$. In this paper, we shall restrict our attention to the case of real-valued $\tau, \nu$ and $z \in E$.

A discrete random variable $x$ is said to have a Borel distribution if it takes the values $1, 2, 3, \cdots$ with the probabilities $\frac{e^{-\lambda}}{1!}, \frac{2\lambda e^{-2\lambda}}{2!}, \frac{9\lambda^2 e^{-3}}{3!}, \cdots$ , respectively, where $\lambda$ is called the parameter.

Elementary distributions such as the Pascal, Poisson, logarithmic, binomial, beta negative binomial and Touchard polynomials have been examined theoretically in Geometric Function Theory.

Very recently, Wanas and Khuttar [2] introduced the Borel distribution (BD) whose probability mass function is

$$P(x = \rho) = \frac{(\rho\lambda)^{\rho-1}e^{-\lambda\rho}}{\rho!}, \quad \rho = 1, 2, 3, \cdots.$$

Wanas and Khuttar introduced a series $\mathcal{M}_\lambda(z)$ whose coefficients are probabilities of the Borel distribution (BD)

$$\mathcal{M}_\lambda(z) = z + \sum_{n=2}^{\infty}\frac{[\lambda(n-1)]^{n-2}e^{-\lambda(n-1)}}{(n-1)!}z^n, (0 < \lambda \leq 1). \tag{6.8}$$

In [19], Murugusundaramoorthy and El-Deeb and Alessa et al [20] defined the Mittag-Leffler-type Borel distribution as follows:

$$\mathcal{P}_\lambda(\tau, \nu; \rho) = \frac{(\lambda\rho)^{\rho-1}}{E_{\tau,\nu}(\lambda\rho)\Gamma(\tau\rho+\nu)}, \quad \rho = 0, 1, 2, \cdots$$

where

$$E_{\tau,\nu}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\tau n+\nu)}, \quad (\tau,\ \nu \in \mathbb{C},\ \Re(\tau) > 0,\ \Re(\nu) > 0).$$

Thus by using (6.7) and (6.8) and by convolution operator, the Mittag-Leffler-type Borel distribution series defined as below

$$B_\lambda(\tau,\nu)(z) = z + \sum_{n=2}^{\infty} \frac{[\lambda(n-1)]![\lambda(n-1)]^{n-2}e^{-\lambda(n-1)}}{(n-1)!E_{\tau,\nu}(\lambda(n-1))\Gamma(\tau(n-1)+\nu)} z^n, (0<\lambda\leq 1).$$

Further, by the convolution operator, we define

$$\begin{aligned} B_\lambda(\tau,\nu)u(z) &= B_\lambda(\tau,\nu)(z) * u(z) \\ &= z + \sum_{n=2}^{\infty} \phi(n)a_n z^n, \quad (\tau,\nu\in\mathbb{C}, \Re(\tau)>0, \Re(\nu)>0, 0<\lambda\leq 1) \end{aligned} \tag{6.9}$$

$$\text{where} \quad \phi(n) = \frac{[\lambda(n-1)]![\lambda(n-1)]^{n-2}\mathrm{e}^{-\lambda(n-1)}}{(n-1)!E_{\tau,\nu}(\lambda(n-1))\Gamma(\tau(n-1)+\nu)} \tag{6.10}$$

$$\text{and} \quad \phi(2) = \frac{\lambda!\mathrm{e}^{-\lambda}}{E_{\tau,\nu}(\lambda)\Gamma(\tau+\nu)}.$$

If $u \in T$ is given by (6.1) then we have

$$B_\lambda(\tau,\nu)u(z) = z - \sum_{n=2}^{\infty} \phi(n)a_n z^n, \tag{6.11}$$

where $\phi(n)$ is given by (1.10).

The study of operators is central to Geometric Function Theory. Analytic function convolutions can be used to express a wide variety of differential and integral operators. This technique has been found to help with upcoming mathematical research and to enhance comprehension of the symmetric and geometric characteristics of these operators. In addition, probability is used in a variety of real-world contexts, including forecasting, insurance, politics, and economics.

## 7 Year wise Schme of the Project

It is evident from the research project's objectives that the study in question is very broad, making it challenging to provide a precise timeline for its completion. Nonetheless, the following is a preliminary outline of the three-year work plan for the current research project's investigation.

The foundational literature pertaining to the core background ideas of the research project will be gathered and examined in the first year, taking into account their applicability to the selected subject. Attending seminars and guest lectures on mathematical analysis and applications is another part of this process.

We will gather the literature on special functions and q-fractional calculus in the second year of the suggested plan, as well as some literature on applications in geometric function theory. This procedure involves gathering research papers and the most recent findings from reputable mathematicians in order to

thoroughly examine the data from a variety of perspectives and draw conclusions.

The third year will see the publication of the new discoveries in reputable referenced journals, as well as discussions and presentations of the new ideas with other researchers working on the held or undertaken research topic. This step also include preparing the research project's documentation for submission to the university in order to be awarded a Ph.D. in Mathematics. Throughout the course of the three-year project, a review of the completed study will be conducted every six months, and the results will be reported to the relevant committee in the form of six monthly progress reports.

## 8 Conclusion

It is claimed that the results of this thesis will be a new basic and original contribution to the subject of Geometric function Theory and applications. It is hoped that the present work will also be useful to scientists, engineers and mathematicians who are engaged in analyzing the universal classical as well as anomalous phenomena governed by univalent functions differential and integral equations to predict and design the future strategy of the study in a better way.

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